NOTE ON A CLASS OF DIFFERENTIAL EQUATIONS OF INFINITE ORDER

BY

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1. Generalizing results by Davis [3] on the Euler differential equation of infinite order, Korobeĭnik [4] proved some theorems for equations of the form

$$\sum_{k=0}^{\infty} z^k h_k(z)\, x^{(k)}(z) = y(z), \tag{1}$$

where the functions $h_k(z)$ $(k=0, 1, \dots)$ and $y(z)$ are analytic in the open unit disc. In [4, Theorem 2] it is required that the functions $h_k(z)$ satisfy the following conditions:

(i) $h_0(z) \equiv 1$,

(ii) $h_k(z) = 1 + \sum_{n=1}^{\infty} h_{kn} z^n$ $(k=1, 2, \dots)$, and there exists a positive sequence $\{h_n\}$ $(n=1, 2, \dots)$ with $\sum_{n=1}^{\infty} h_n < 1$, such that $|h_{kn}| \leqslant h_n$ $(n=1, 2, \dots)$ for all sufficiently large k.

As noted by Korobeĭnik, the latter conditions are rather similar to those occurring in a theorem by Boas on bases in spaces of analytic functions, cf. [2, Theorem 4.1]. The purpose of this note is to obtain a result about (1) directly from Boas's theorem (in a form given by Arsove [1]).

2. Let N_1 denote the open unit disc of the complex plane C. Let $A(N_1)$ be the complex linear space of functions analytic in N_1, with algebraic operations defined pointwise, and with the topology of uniform convergence on compact subsets of N_1. Let $I[\varrho, \sigma]$ denote the class of entire functions of growth at most order ϱ, type σ.

A sequence $\{A_n\}$ $(n=0, 1, \dots)$ in $A(N_1)$ is called a basis if every $y \in A(N_1)$ has a unique expansion

$$y = \sum_{n=0}^{\infty} y_n A_n, \quad y_n \in C \quad (n=0, 1, \dots), \tag{2}$$

where the series converges in $A(N_1)$. The simplest example is, of course, the system $\{e_n\}$ $(n=0, 1, \dots)$, where $e_n(z) = z^n$. A basis $\{A_n\}$ in $A(N_1)$ is called proper if there exists an automorphism T on $A(N_1)$ such that $Te_n = A_n$ $(n=0, 1, \dots)$.

In this case the coefficients in (2) satisfy

$$\limsup_{n\to\infty} |y_n|^{1/n} \leq 1, \tag{3}$$

and, conversely, to each complex sequence $\{y_n\}$ $(n=0, 1, \ldots)$ satisfying (3) there exists a unique $y \in A(N_1)$ with expansion (2). (This property may be used as an alternative definition of a proper basis.) A basis $\{A_n\}$ is called a Pincherle basis if it has the form

$$A_n(z) = z^n(1+\lambda_n(z)) \quad (n=0, 1, \ldots) \tag{4}$$

where $\lambda_n \in A(N_1)$, $\lambda_n(0)=0$. We recall the following result from ARSOVE [1, Corollary 9.1]. If $\lambda_n \in A(N_1)$, $\lambda_n(z) = \sum_{k=1}^{\infty} \lambda_{nk} z^k$ $(n=0, 1, \ldots)$ and

$$\limsup_{n\to\infty} \sum_{k=1}^{\infty} |\lambda_{nk}| r^k < 1$$

for every r, $0<r<1$, then (4) is a proper Pincherle basis in $A(N_1)$.

3. For a function $x(z) = \sum_{n=0}^{\infty} x_n z^n$ in $A(N_1)$ we may write the left-hand side of (1) in the form

$$\sum_{k=0}^{\infty} z^k h_k(z) \sum_{n=k}^{\infty} \frac{n!\, x_n z^{n-k}}{(n-k)!}.$$

Not bothering about questions of convergence we obtain after rearrangement

$$\sum_{n=0}^{\infty} x_n n!\, z^n \sum_{k=0}^{n} \frac{h_k(z)}{(n-k)!}, \tag{5}$$

a formula which suggests the connection of Pincherle bases with the problem of solving (1). We prove the following

THEOREM. *Let $h_n(z) = \sum_{k=0}^{\infty} h_{nk} z^k$ be analytic in N_1 $(n=0, 1, \ldots)$. Let*

$$\alpha_n = \sum_{k=0}^{n} \frac{h_{k0}}{(n-k)!} \neq 0 \quad (n=0, 1, \ldots), \tag{6}$$

and assume that there exist positive numbers ϱ and σ such that

$$\lim_{n\to\infty} \{n!^{1-1/\varrho} |\alpha_n|\}^{1/n} = (\varrho\sigma)^{-1/\varrho}. \tag{7}$$

Finally assume that

$$\limsup_{n\to\infty} \frac{1}{|\alpha_n|} \sum_{k=0}^{n} \sum_{l=1}^{\infty} \frac{|h_{kl}| r^l}{(n-k)!} < 1 \tag{8}$$

for every r, $0<r<1$. Then for each $y \in A(N_1)$ there exists a unique $x \in I[\varrho, \sigma]$ satisfying (1) for all $z \in N_1$.

PROOF. An entire function $x(z)=\sum_{n=0}^{\infty} x_n z^n$ belongs to $I[\varrho, \sigma]$ if and only if

$$\limsup_{n\to\infty} \{n!^{1/\varrho}|x_n|\}^{1/n} \leqslant (\varrho\sigma)^{1/\varrho}. \tag{9}$$

Hence, from (7) and (8) we obtain that for such a function the left-hand side of (1) is equal to the expression (5) if $z \in N_1$, and even that (5) converges uniformly on compact subsets of N_1.

Now let $y \in A(N_1)$. Since (8) implies that the system

$$\Lambda_n(z)=z^n\left\{1+\frac{1}{\alpha_n}\sum_{k=0}^{n}(h_k(z)-h_{0k})\right\} \quad (n=0, 1, \ldots)$$

is a proper Pincherle basis for $A(N_1)$, there exists a unique expansion $y=\sum_{n=0}^{\infty} y_n\Lambda_n$, where the sequence $\{y_n\}$ satisfies (3). So for $x \in I[\varrho, \sigma]$ to satisfy (1) in N_1 we should have

$$\alpha_n n!x_n = y_n \quad (n=0, 1, \ldots). \tag{10}$$

The condition (6) implies that the coefficients $\{x_n\}$ $(n=0, 1, \ldots)$ are uniquely determined by (10), while (3) and (7) imply that this sequence has the property (9). This completes the proof.

The following corollary still contains Korobeĭnik's result.

COROLLARY. *Let* $h_k(z)=1+\sum_{n=1}^{\infty} h_{kn}z^n$ $(k=0, 1, \ldots)$ *and assume*

$$\limsup_{n\to\infty} \sum_{l=1}^{\infty} |h_{nk}|r^l<1$$

for every r, $0<r<1$. *Then for each* $y \in A(N_1)$ *there exists a unique* $x \in I[1, 1]$ *satisfying* (1) *for every* $z \in N_1$.

PROOF. In this case $\alpha_n=\sum_{k=0}^{n}(n-k)!^{-1}$, so (6) holds, and (7) is satisfied with $\varrho=\sigma=1$. Thus we are left with the task of checking (8).

Let $0<r<1$. By assumption there exist q, $0<q<1$, and a positive integer n_0 such that

$$\sum_{n=1}^{\infty}|h_{nl}|r^l<q \quad (n>n_0).$$

If $n>n_0$, then

$$\frac{1}{|\alpha_n|}\sum_{k=0}^{n}\sum_{l=1}^{\infty}\frac{|h_{kl}|r^l}{(n-k)!} \leqslant \sum_{k=0}^{n_0}\frac{1}{(n-k)!}\sum_{l=1}^{\infty}|h_{kl}|r^l+\frac{q}{\alpha_n}\sum_{k=n_0+1}^{n}\frac{1}{(n-k)!}.$$

The first term on the right tends to zero as $n\to\infty$, and the second one is dominated by q. Thus (8) holds, whence the result.

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